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SCATTERING NUMBER AND EXTREMAL NON-HAMILTONIAN GRAPHS

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The scattering number, $s(G)$, of a graph G is defined by $s(G)=\max\{k(G-X)-|X|:X\subseteq V(G),\ k(G-X)\neq 1\}$, where $k(H)$ denotes the number of components of the graph H. Let Q_0, Q_1 and Q_2 denote the words traceable, hamiltonian and hamiltonian-connected, respectively. It is shown that, for $i=0$, 1 or 2, if G is an (n, q) graph with $n\geqslant 19+i$ and $q\geqslant\binom{n-3}{2}+3i+3$ such that $s(G)\leqslant 1-i$ then G is Q_i unless G is a certain exceptional graph. A more general conjecture, which is proved in a special case, is proposed and a related problem is stated.

1. Introduction and notation

Notation. Let Q_0, Q_1 and Q_2 denote the words traceable, hamiltonian and hamiltonian-connected, respectively.

The following extremal result was proved by or can be deduced from results proved by Bondy [2] and Ore [10, 11]:

Theorem 1.1. *Let $i=0$, 1 or 2 and let G be an (n, q) graph with $n\geqslant i+2$ and $q\geqslant\binom{n-1}{2}+i$. Then either G is Q_i or $G\cong K_i+(K_1\cup K_{n-i-1})$ or $n=i+4$ and $G\cong K_{i+1}+\bar{K}_3$.*

Following Jung [7], we define the *scattering number*, $s(G)$, of a graph G by

$$s(G)=\max\{k(G-X)-|X|:X\subseteq V(G),\qquad k(G-X)\neq 1\},$$

where $k(H)$ denotes the number of components of the graph H. For any graph G the condition $s(G)\leqslant 0$ is equivalent to the condition $t(G)\geqslant 1$, where $t(G)$ is the toughness of G (see [5]).

For $i=0$, 1 or 2, the obvious necessary condition, $s(G)\leqslant 1-i$, for G to be Q_i fails if G is one of the extremal graphs for Theorem 1.1. Therefore the problem of determining the maximum size of an n-vertex graph G with $s(G)\leqslant 1-i$ which is not Q_i is suggested. We solve this problem for n sufficiently large in Theorem 3.1 and propose the following more general conjecture which we prove for the pair $(h, i)=(1, 0)$ in Theorem 4.1. Our methods do not appear to be sufficiently powerful to tackle the conjecture in general.

Conjecture 1.2. Let h and i be integers with $h \geq 0$ and $0 \leq i \leq 2$. There exists an integer $N(h, i)$ such that if G is a graph with order $n \geq N(h, i)$, size $q \geq \binom{n-2h-3}{2} + (2h+3)(h+i+1)$ and scattering number $s(G) \leq 1-h-i$ then either G is Q_i or $G \cong K_{h+i} + A_{n-3h-i-3,2h+3}$ where, for $p \geq r$, $A_{p,r}$ denotes the graph of order $p+r$ obtained by identifying an end vertex of each edge of rK_2 with a distinct vertex of K_p (see Fig. 1).

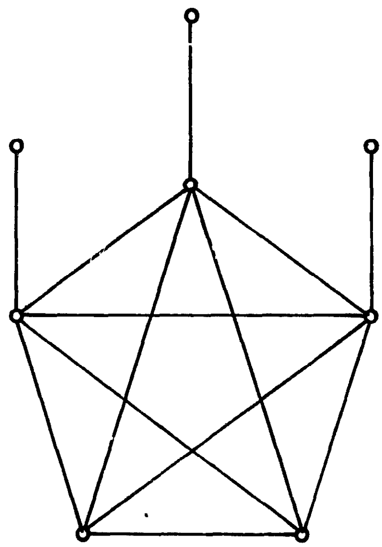

Fig. 1. The graph $A_{5,3}$.

To get an impression of how strong this size condition is, we observe that it follows from the results of [4] that if G has size greater than $\binom{n-h-2}{2} + (h+2)(h+i+1)$ and $\delta(G) \geq h+i+1$ (which is satisfied if $s(G) \leq 1-h-i$) then, provided that G has large enough order, G is Q_i. We refer the reader to [1] for any graph-theoretic terminology which is not explicitly defined. By an (n, q) graph we mean a graph with n vertices and q edges.

2. Preliminaries

In this section, we quote some results from the literature and prove some simple propositions which are needed to prove our main result of Section 3.

Definition. [3]. A property Q is said to be *k-stable* if whenever $G+uv$ has property Q and $\deg_G(u) + \deg_G(v) \geq k$ then G itself has property Q.

Theorem 2.1. [3]. *For $i = 0$, 1 or 2, Q_i is $(n+i-1)$-stable.*

Theorem 2.2. [9, 12]. *Let $i = 0$, 1 or 2 and let G be a graph of order $n \geq 3$ which is not Q_i. Then there exists an integer k, with $i \leq k \leq \frac{1}{2}(n+i-2)$, such that G has at least $k-i+1$ vertices of degree less than or equal to k.*

Proposition 2.3. *Let G be a non-Q_2 $(n, \binom{n-1}{2}+1)$ graph, where $n \geq 9$. Then G is an edge-deleted subgraph of $K_2+(K_1 \cup K_{n-3})$.*

Proof. If G has a vertex x of degree 1 or 2 then $G-x \cong K_{n-1}$ or $K_{n-1}-e$ and so G is an edge-deleted subgraph of $K_2+(K_1 \cup K_{n-3})$. Therefore suppose that $\delta(G) \geq 3$. Since G is not Q_2, Theorem 2.2 implies the existence of an integer k, $3 \leq k \leq \frac{1}{2}n$, such that G has at least $k-1$ vertices of degree less than or equal to k. Therefore $\binom{n-1}{2}+1=|E(G)| \leq k(k-1)+\binom{n-k+1}{2}$ and so

(1) $\quad (2k-4)n \leq 3k^2-3k-4.$

Note that $2k-4$ is positive. Since $n \geq 2k$ it follows from (1) that $1 \leq k \leq 4$. On the other hand, since $n \geq 9$, it follows from (1) that $k \leq 2$ or $k \geq 5$. Since $k \geq 3$, we have a contradiction. □

Definition. A graph G is *k-edge-hamiltonian* if every linear forest of length k is contained in a hamiltonian cycle of G. A graph G is *kK_2-hamiltonian* if every set of k independent edges is contained in a hamiltonian cycle of G.

Theorem 2.4 [8]. *Let G be a graph of order n and k an integer, $0 \leq k \leq n-2$. The following condition is sufficient for G to be k-edge-hamiltonian: for all integers j with $k+1 \leq j \leq \frac{1}{2}(n+k-1)$, G has at most $j-k-1$ vertices of degree less than or equal to j.*

Proposition 2.5. *Let $k=2$ or 3 and let G be an (n, q) graph with $n \geq k+8$ and $q \geq \binom{n-1}{2}+k$ which is not k-edge-hamiltonian. Then $\delta(G) \leq k+1$.*

Proof. By Theorem 2.4, there exists an integer j with $k+1 \leq j \leq \frac{1}{2}(n+k-1)$ such that G has at least $j-k$ vertices of degree less than or equal to j. If $j=k+1$ the result follows. Therefore suppose that $j \geq k+2$. Therefore $\binom{n-1}{2}+k \leq q \leq j(j-k)+\binom{n-j+k}{2}$ and so

(1) $\quad (2j-2k-2)n \leq 3j^2+(1-4k)j+k^2-3k-2.$

Since $k+2 \leq j$, $2j-2k-2$ is positive. Since $j \leq \frac{1}{2}(n+k-1)$, $n \geq 2j-k+1$ and so, substituting in (1), we deduce that $k \leq j \leq k+3$. On the other hand, since $n \geq k+8$, it follows from (1) that $j \leq k+1$ or $j \geq k+4$. Since $j \geq k+2$, we have a contradiction. □

Proposition 2.6. *Let G be an (n, q) graph with $n \geq 10$ and $q \geq \binom{n-1}{2}+2$ which is not $2K_2$-hamiltonian. Then $G \cong K_2+(K_1 \cup K_{n-3})$.*

Proof. By Proposition 2.5, $\delta(G) \leq 3$. So $\delta(G)=2$ or 3 and G is $K_3+(K_1 \cup K_{n-4})$ or an edge-deleted subgraph thereof. It is straightforward to verify that all such graphs are $2K_2$-hamiltonian except $K_2+(K_1 \cup K_{n-3})$. □

Proposition 2.7. *Let G be an (n, q) graph with $n \geqslant 11$ and $q \geqslant \binom{n-1}{2}+3$. Then G is $3K_2$-hamiltonian.*

Proof. By Proposition 2.5, $\delta(G)=3$ or 4 and so G is $K_4+(K_1 \cup K_{n-5})$ or an edge-deleted subgraph thereof. It is straightforward but tedious to verify that all such graphs are $3K_2$-hamiltonian. □

3. Conjecture 1.2; the case $h=0$

Theorem 3.1. *For $i=0$, 1 or 2, let G be a graph with order $n \geqslant 19+i$, size $q \geqslant \binom{n-3}{2}+3i+3$ and scattering number $s(G) \leqslant 1-i$ which is not Q_i. Then $G \cong K_i + A_{n-i-3,3}$.*

Proof. We may assume w.l.o.g. that G is maximally non-Q_i. In particular, it follows since $s(G) \leqslant 1-i$ that

(1) $\delta(G) \geqslant i+1$

and, by Theorem 2.1, that

(2) if u and v are nonadjacent then $\deg(u)+\deg(v) \leqslant n+i-2$.

(3) Either G has two vertices of degree $i+1$ or G has two vertices of degree $i+2$ and a third of degree $i+1$ or $i+2$.

Proof of (3). Since G is not Q_i, Theorem 2.2 implies the existence of an integer k, $i \leqslant k \leqslant \frac{1}{2}(n+i-2)$, such that G has at least $k-i+1$ vertices of degree less than or equal to k. If $k \leqslant i+2$ then the result follows due to (1). Therefore suppose that $k \geqslant i+3$. Therefore $\binom{n-3}{2}+3i+3 \leqslant q \leqslant k(k-i+1)+\binom{n-k+i-1}{2}$ and so

(3.1) $(2k-2i-4)n \leqslant 3k^2+(5-4i)k+i^2-9i-16.$

Since $k \geqslant i+3$, $2k-2i-4$ is positive. Since $k \leqslant \frac{1}{2}(n+i-2)$, we have $n \geqslant 2k-i+2$. Substituting for n in (3.1), we deduce that $i+1 \leqslant k \leqslant i+8$. On the other hand, since $n \geqslant 19+i$, it follows from (3.1) that $k \leqslant 1+2$ or $k \geqslant i+9$. Since $k \geqslant i+3$, we have a contradiction which completes the proof of (3).

(4) If G has at least three vertices x, y and z of degree less than or equal to $i+2$ then (a) $G-x-y-z \cong K_{n-3}$ and (b) if $u \in V(G-x-y-z)$ is adjacent to one of x, y and z then u is adjacent to each of x, y and z which has degree $i+2$.

Proof of (4).

(a) Let $H=G-x-y-z$ and suppose that u and v are nonadjacent vertices of H. H has at least $\binom{n-3}{2}-3$ edges and so $\delta(H) \geqslant n-4$. Therefore $\deg_G(u)+\deg_G(v) \geqslant 2n-8 \geqslant n+i+11$, which contradicts (2).

(b) Since $H \cong K_{n-3}$, $\deg_G(u) \geq n-3$. If $\deg_G(x) = i+2$ then $ux \in E(G)$: for otherwise, we contradict (2). The result follows.

(5) Suppose $i = 0$. Then $G \cong A_{n-3,3} \cong K_0 + A_{n-3,3}$.

Proof of (5). By (3) either (a) G has two vertices x and y of degree 1 or (b) G has two vertices x and y of degree 2 and a third vertex z of degree 1 or 2.

(a) Since $s(G) \leq 1$, $xy \notin E(G)$ and $N(x) \neq N(y)$. Suppose that xu and yv are edges. Since G is not Q_0, there is no hamiltonian u–v path in $G-x-y$. So $G-x-y$ is a non-Q_2 graph of order $n-2$ with at least $\binom{n-3}{2}+1$ edges. By Theorem 1.1 and Proposition 2.3, G is $K_2 + (K_1 \cup K_{n-3})$ or an edge-deleted subgraph thereof. If $\delta(G-x-y) = 2$ then u and v are the neighbours of the vertex of degree two in $G-x-y$, in which case we have a contradiction of $s(G) \leq 1$ because $k(G-u-v) \geq 4$. Therefore $G-x-y \cong K_1 + (K_1 \cup K_{n-4})$. If $\deg_{G-x-y}(u) = n-3$ then $k(G-u) \geq 3$ which again contradicts $s(G) \leq 1$. Therefore u and v each have degree $n-4$ in $G-x-y$ and consequently $G \cong A_{n-3,3}$.

(b) By (4), $G-x-y-z \cong K_{n-3}$ and x and y are adjacent to the same set N of vertices in $G-x-y-z$ and $N(z) \cap V(G-x-y-z) \subseteq N$. If $N = \emptyset$ then $G \cong K_{n-3} \cup K_3$, which contradicts $s(G) \leq 1$. If $N = \{u, v\}$ then, since G is not Q_0, $N(z) \subseteq \{u, v\}$ and $k(G-u-v) \geq 4$, which again contradicts $s(G) \leq 1$. Therefore suppose that $N = \{u\}$. Since G is not Q_0, at most one of the edges xy, yz and zx is in G. Consequently xy and zu are edges of G and $k(G-u) \geq 3$, contrary to $s(G) \leq 1$.

(6) Suppose $i = 1$ or 2 and $\Delta(G) \leq n-3$. Then G has two vertices of degree $i+1$.

Proof of (6). If G does not have two vertices of degree $i+1$ then by (3), G has two vertices x and y of degree $i+2$ and a vertex z of degree $i+1$ or $i+2$. Since $i \geq 1$, we may assume that x is adjacent to some vertex u of $G-x-y-z$. By (4), $yu \in E(G)$ and $G-x-y-z \cong K_{n-3}$. Therefore $\deg(u) \geq n-2$, which contradicts $\Delta(G) \leq n-3$.

(7) Suppose $i = 1$. Then $\Delta(G) \geq n-2$.

Proof of (7). Suppose to the contrary that $\Delta(G) \leq n-3$. By (6), G has two vertices x and y of degree 2.

First suppose $xy \in E(G)$. Since $s(G) \leq 0$, we may assume that xu and yv are edges of G. So $G-x-y$ has $n-2$ vertices and at least $\binom{n-3}{2}+3$ edges and hence, by Theorem 1.1, $G-x-y$ is Q_2. Therefore $G-x-y$ has a hamiltonian u–v path and G is Q_1, which is a contradiction.

Now suppose $xy \notin E(G)$. If $N(x) = N(y)$ then $k(G-N(x)) \geq 3$, which contradicts $s(G) \leq 0$. If $N(x) = \{u, z\}$ and $N(y) = \{v, z\}$ then $G-x-y-z$ has

$n-3$ vertices and, since $\Delta(G) \leq n-3$, at least $\binom{n-4}{2}+3$ edges. By Theorem 1.1, $G-x-y-z$ is Q_2 and so contains a hamiltonian u–v path. Therefore G is Q_1, which is a contradiction. Therefore we may assume that $N(x)=\{t, u\}$ and $N(y)=\{v, w\}$. Since G is maximally non-Q_1, tu and vw are edges but there is no hamiltonian cycle in $G-x-y$ containing them both. Since $G-x-y$ has $n-2$ vertices and at least $\binom{n-3}{2}+2$ edges, it follows from Proposition 2.6 that $G-x-y \cong K_2+(K_1 \cup K_{n-5})$. Therefore w.l.o.g. t and u are the vertices of the K_2 in $G-x-y$ and so have degree exceeding $n-3$ in G, which is a contradiction.

(8) Suppose $i=1$ or 2. Then G has no vertex of degree $n-2$.

Proof of (8). If $\deg(x)=n-2$ and $xy \notin E(G)$ then, by (1), $\deg(x)+\deg(y) \geq n+i-1$, which contradicts (2).

(9) Suppose $i=1$. Then $G \cong K_1+A_{n-4,3}$.

Proof of (9). By (7) and (8), $\Delta(G)=n-1$. Suppose that $\deg(u)=n-1$ and consider $G-u$ which has $n-1 \geq 19$ vertices and at least $\binom{n-4}{2}+3$ edges. Since G is not Q_1, $G-u$ is not Q_0. Since $s(G) \leq 0$, $s(G-u) \leq 1$. By (5), $G-u \cong A_{n-4,3}$ and so $G \cong K_1+A_{n-4,3}$.

(10) Suppose $i=2$. Then $\Delta(G)=n-1$.

Proof of (10). Suppose that $\Delta(G) \neq n-1$. By (8), $\Delta(G) \leq n-3$ and so, by (6), G has two vertices x and y of degree 3. Since $s(G) \leq -1$, $N(x) \cup \{x\} \neq N(y) \cup \{y\}$. $G-x-y$ has $n-2$ vertices and at least $\binom{n-3}{2}+3$ edges. By Theorem 1.1 and Proposition 2.7, $G-x-y$ is Q_2 and $3K_2$-hamiltonian. Suppose that u and v are vertices of $G-x-y$ which are adjacent to x or y. Then $G-x-y-u$ has $n-3$ vertices and at least $\binom{n-4}{2}+3$ edges and so, by Theorem 1.1 and Proposition 2.6, is Q_2 and $2K_2$-hamiltonian. By (2) and the fact that $\Delta(G) \leq n-3$, u and v are together incident with at most $2n-7$ edges of G. Therefore $G-x-y-u-v$ has $n-4$ vertices and at least $\binom{n-5}{2}+3$ edges and so, by Theorem 1.1, is Q_2. Using this information it is fairly simple to show that, regardless of which vertices are adjacent to x and y, G is Q_2, which is contrary to hypothesis.

(11) Suppose $i=2$. Then $G \cong K_2+A_{n-5,3}$.

Proof of (11). By (10), $\Delta(G)=n-1$. Suppose that $\deg(u)=n-1$ and consider $G-u$ which has $n-1 \geq 20$ vertices and at least $\binom{n-4}{2}+6$ edges. Since G is not Q_2, $G-u$ is not Q_1. Since $s(G) \leq -1$, $s(G-u) \leq 0$. By (9), $G-u \cong K_1+A_{n-5,3}$ and so $G \cong K_2+A_{n-5,3}$.

Statements (5), (9) and (11) complete the proof of Theorem 3.1. □

4. Conjecture 1.2; the case $h=1$, $i=0$

Theorem 4.1. *Let G be a graph with order $n \geqslant 31$, size $q \geqslant \binom{n-5}{2}+10$ and scattering number $s(G) \leqslant 0$ which is not traceable. Then $G \cong K_1 + A_{n-6,5}$.*

Proof. We may assume w.l.o.g. that G is maximally non-traceable. It follows since $s(G) \leqslant 0$ that

(1) G is 2-connected

and, by Theorem 2.1, that

(2) if u and v are nonadjacent then $\deg(u)+\deg(v) \leqslant n-2$.

Notation. Let $D(G)$ denote the degree sequence of G in nondecreasing order.

(3) $D(G)$ is majorized by one of the sequences:

$$2,2,2,n-4,n-4,n-4,\dots,n-4,n-4,n-4,n-1,n-1;$$
$$3,3,3,\quad 3,n-5,n-5,\dots,n-5,n-5,n-1,n-1,n-1;$$
$$4,4,4,\quad 4,\quad 4,n-6,\dots,n-6,n-1,n-1,n-1,n-1.$$

Proof of (3). By (1) and Corollary 1.1 of [4], there exists an integer k, $2 \leqslant k \leqslant \frac{1}{2}(n-2)$, such that $D(G)$ is majorized by the degree sequence of $K_k + (\bar{K}_{k+1} \cup K_{n-2k-1})$. If $k=2$, 3 or 4 then the result follows. Therefore suppose that $k \geqslant 5$. Since $\binom{n-5}{2}+10 \leqslant q \leqslant (k+1)k + \binom{n-k-1}{2}$, we deduce that $(2k-8)n \leqslant 3k^2+5k-48$ and hence obtain a contradiction using the facts that $n \geqslant 2k+2$ and $n \geqslant 31$.

(4) G has no vertex u with $\frac{1}{2}(n-2) \leqslant \deg(u) \leqslant n-7$ or $n-3 \leqslant \deg(u) \leqslant n-2$.

Proof of (4). If $\deg(u)=n-3$ or $n-2$ and $uv \notin E(G)$ then, by (1), $\deg(v) \geqslant 2$ and we have a contradiction of (2). Now suppose that $\deg(u)=t$ with $\frac{1}{2}(n-2) \leqslant t \leqslant n-7$. By (2), the $n-t-1$ vertices not adjacent to u each have degree not exceeding $n-t-2$. Let $k=n-t-2$. Then $5 \leqslant k \leqslant \frac{1}{2}(n-2)$ and G has at least $k+1$ vertices of degree not exceeding k and we obtain a contradiction as in the proof of (3).

Notation. Let $A=\{v \in V(G): \deg(v) \geqslant n-6\}$ and $B=\{v \in V(G): \deg(v) \leqslant \frac{1}{2}(n-3)\}$. So $V(G)=A \cup B$. Let $H=\langle B \rangle$.

(5) $\langle A \rangle$ is complete.

Proof of (5). If $u, v \in A$ then $\deg(u)+\deg(v) \geqslant 2n-12 > n-2$ and so, by (2), $uv \in E(G)$.

(6) Every vertex of G has degree 2, 3, 4, $n-6$, $n-5$, $n-4$ or $n-1$.

Proof of (6). Suppose that $u \in V(G)$ with $5 \leqslant \deg(u) \leqslant \frac{1}{2}(n-3)$. By (2), u is adjacent to every vertex of A. Therefore $|A| \leqslant \deg(u) \leqslant \frac{1}{2}(n-3)$ and so, by (3), $D(G)$ is majorized by the sequence 4, 4, 4, $\lfloor (n-3)/2 \rfloor, \ldots, \lfloor \frac{1}{2}(n-3) \rfloor$, $n-4, \ldots, n-4$, $n-1$, $n-1$, $n-1$, $n-1$ which contains $\lceil \frac{1}{2}(n-3) \rceil$ terms equal to $\lfloor \frac{1}{2}(n-3) \rfloor$ and $\lfloor \frac{1}{2}(n-11) \rfloor$ terms equal to $n-4$. Since $q \geqslant \binom{n-5}{2} + 10$ and $n \geqslant 31$, we obtain a contradiction from which we deduce that no such vertex u exists. The result now follows by (4).

(7) $K_{n-3} \nsubseteq G$.

Proof of (7). Suppose $K_{n-3} \subseteq G$. Since $\delta(G) \geqslant 2$, $q \geqslant \binom{n-3}{2} + 3$. Therefore by Theorem 3.1, $G \cong A_{n-3,3}$ which contradicts $s(G) \leqslant 0$.

(8) $K_{n-4} \nsubseteq G$.

Proof of (8). Suppose $K_{n-4} \subseteq G$ and let w, x, y and z be the other vertices. By (2), (5), (6) and (7), vertices w, x, y and z each have degree 2 or 3. So $A = V(K_{n-4})$ and $B = \{w, x, y, z\}$.

$K_{1,3} \nsubseteq H$. For if wx, wy and wz are edges then, by (2), there can be no edge from x, y or z to A, which contradicts $s(G) \leqslant 0$.

$P_4 \nsubseteq H$. For suppose that wx, xy and yz are edges. Since G is not traceable, there is no edge from w or z to A and so $wz \in E(G)$ and there is no edge from A to B, which contradicts $s(G) \leqslant 0$.

$2P_2 \nsubseteq H$. For if wx and yz are edges then $H \cong 2P_2$ and, since $s(G) \leqslant 0$, there exist an edge from A to $\{w, x\}$ and an independent edge from A to $\{y, z\}$ and we have a contradiction because G is not traceable.

$P_3 \nsubseteq H$. For suppose that wx and xy are edges. Then $H \cong P_3 \cup K_1$ or $K_3 \cup K_1$ and so z has at least two neighbours in A. Since G is not traceable there is no edge from w or y to A and so $wy \in E(G)$ and there is no edge from x to A, which contradicts $s(G) \leqslant 0$.

$H \cong \bar{K}_4$. For if $wx \in E(G)$ then $H \cong K_2 \cup \bar{K}_2$ and G contains edges zu, zv and yt, where t may be u or v. Since $s(G) \leqslant 0$, w or x is adjacent to some vertex of $A - \{t\}$. Since $s(G) \leqslant 0$, w or x is adjacent to some vertex of $A - \{t\}$. Since G is not traceable, it follows that there is no edge from w, x or y to $A - \{u, v\}$, which contradicts $s(G) \leqslant 0$.

By (6), each vertex of A is adjacent to 0, 1 or 4 vertices of B. Since G is not traceable, there exists a vertex v of A adjacent to w, x, y and z. Since $s(G) \leqslant 0$, we may assume that there exist four independent edges from B to $A - \{v\}$, in which case we have a contradiction because G is not traceable.

(9) $|A| = n-5$.

Proof of (9). If $|A| \leqslant n-6$ then, by (6), G has at least six vertices of degree not exceeding 4 and so $q \leqslant 24 + \binom{n-6}{2}$. Since this contradicts $n \geqslant 31$ and $q \geqslant \binom{n-5}{2} + 10$, $|A| \geqslant n-5$. The result follows by (5) and (8).

Notation. Let $B = \{v, w, x, y, z\}$ and let $\{\ldots p, q, r, s, t, u\}$ be a subset of A.

(10) If $V \subseteq B$ is the union of the vertex sets of $k \geqslant 1$ components of H then there are at least $k+1$ edges from V to A.

Proof of (10). This follows since $s(G) \leqslant 0$.

(11) $K_{1,4} \nsubseteq H$. For if v is adjacent to w, x, y and z then, by (2) and (6), there are no edges from A to B, which contradicts (10).

(12) $C_5 \nsubseteq H$. For if $C_5 \subseteq H$ then, since G is not traceable, there are no edges from A to B, which contradicts (10).

(13) $P_5 \nsubseteq H$. For suppose that vw, wx, xy and yz are edges. Since G is not traceable, we deduce using (11) and (12) that vy and wz are edges, that $H \cong K_{2,3}$ and that there are no edges from v, x and z to A. But now $k(G-w-y) \geqslant 4$, which contradicts $s(G) \leqslant 0$.

(14) $C_4 \nsubseteq H$. For if vw, wx, xy and yv are edges then, by (13), z is not adjacent to v, w, x and y and so, by (10), G is traceable, which is a contradiction.

(15) $P_4 \nsubseteq H$. For suppose that vw, wx and xy are edges. If $vx \in E(G)$ then, by (11), (13) and (14), $\deg_H(y) = 1$ and $\deg_H(z) = 0$ and so there are two edges from z to A and an edge from Y to A and G is traceable. Therefore suppose that vx and wy are not edges. Since G is not traceable we may assume w.l.o.g. that xz, uv, uy and uz are the only other edges of G incident with v, y or z. But now $k(G-u-x) \geqslant 3$, which contradicts $s(G) \leqslant 0$.

(16) $K_3 \nsubseteq H$. For suppose that vw, wx and xv are edges. By (15), $H \cong K_3 \cup K_2$ or $K_3 \cup \bar{K}_2$ depending on whether or not yz is an edge. By (10), we may assume w.l.o.g. in the first case that zu and xt are edges and in the second case that zu, zt, ys and xr are edges. In either case, we have a contradiction because G is not traceable.

(17) $K_{1,3} \nsubseteq H$. For suppose that vw, vx and vy are edges. By (11) and (15), $H \cong K_{1,3} \cup K_1$. By (10), we may assume that zu and zt are edges. Considering the vertices t, u and v, it follows from (2) that $\deg_G(v) = 3$. By (10), we may assume that $ys \in E(G)$. Since G is not traceable, it follows that w, x and y are each adjacent to v and s and to no other vertex. But now $k(G-s-v) \geqslant 4$, which contradicts $s(G) \leqslant 0$.

(18) $2K_2 \nsubseteq H$. For suppose that vw and xy are edges. By (15) and (16), we may assume w.l.o.g. that $H \cong P_3 \cup K_2$ or $2K_2 \cup K_1$ depending on whether or not yz is an edge. By (10), we may assume w.l.o.g. in the first case that zu and wt are edges and in the second case that zu, zt, ys and wr are edges. In either case we have a contradiction because G is not tracable.

(19) $P_3 \nsubseteq H$. For suppose that vw and wx are edges. By (16), (17) and (18),

$H \cong P_3 \cup \bar{K}_2$ and so, by (10), we may assume that zu, zt and ys are edges. Since G is not traceable, v and x are each adjacent to s and to no other vertex of A. By (10), y is adjacent to a vertex of $A-\{s\}$ and so G is traceable, which is a contradiction.

(20) $H \cong \bar{K}_5$. For suppose that vw is an edge. By (18) and (19), $H \cong K_2 \cup \bar{K}_3$. By (10), we may assume that vu, wt, xs, yr and zq are edges. But now, regardless of which vertex of $A-\{q\}$ is adjacent to z, G is traceable, which is a contradiction.

We can now complete the proof of Theorem 4.1 by showing that $G \cong K_1 + A_{n-6,5}$. If no vertex in A is adjacent to more than one vertex of B then clearly G is traceable, which is a contradiction. Therefore we may assume w.l.o.g. that uv and uw are edges. By (10) and (20), we may assume that vt, ws, xr, yq and zp are edges. Since $\delta(G) \geq 2$ and G is not traceable, u is also adjacent to x, y and z and so $G \cong K_1 + A_{n-6,5}$. □

By using the appropriate results from the literature, an argument analogous to that used in the proof of Theorem 4.1 (1)–(6) can be developed for general values of h and i, but not the argument beyond that point.

5. A related problem

Let $l(G)$ denote the minimum number of vertex disjoint paths needed to cover the vertices of G. For any graph G, let

$$r(G) = \max\{l(G-X) - |X| : X \subseteq V(G), \quad l(G-X) \neq 1\}.$$

The conditions $r(G) \leq 0$ and $s(G) \leq 0$ are both obvious necessary conditions for G to be hamiltonian. Clearly $r(G) \leq 0$ implies $s(G) \leq 0$. We showed in Theorem 3.1 that, for $n \geq 20$, the maximum size of an n-vertex nonhamiltonian graph G such that $s(G) \leq 0$ is $\binom{n-3}{2} + 6$.

Problem 5.1. Determine the maximum size $f(n)$ of an n-vertex nonhamiltonian graph G such that $r(G) \leq 0$.

Let H denote the graph shown in Fig. 2 of [6] and let T be a triangle in H which contains a vertex of degree two. The graph obtained by identifying T with a triangle of K_{n-7} shows that $f(n) \geq \binom{n-7}{2} + 17$, for $n \geq 10$. We conjecture that there exist integers N and c such that $f(n) \leq \binom{n-7}{2} + c$ for $n \geq N$.

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